

Three Loop Free Energy Using Screened Perturbation Theory ^a

Jens O. Andersen

Physics Department, Ohio State University, Columbus OH 43210, USA

The conventional weak-coupling expansion for the pressure of a hot plasma shows no sign of convergence unless the coupling constant g is tiny. In this talk, I discuss screened perturbation theory (SPT) which is a reorganization of the perturbative expansion by adding and subtracting a local mass term in the Lagrangian. We consider several different mass prescriptions, and compute the pressure to three-loop order. The SPT-improved approximations appear to converge for rather large values of the coupling constant.

1 Introduction

The heavy-ion collision experiments at RHIC and LHC give us for the first time the possibility to study the properties of the high-temperature phase of QCD. There are many methods that can be used to calculate the properties of the quark-gluon plasma. One of these methods is lattice gauge theory, which gives reliable results for equilibrium properties such as the pressure but cannot easily be applied to real-time processes. Another method is the weak-coupling expansion, which can be applied to both static and dynamical quantities. However, it turns out that the weak-coupling expansion for e.g. the pressure does not converge unless the strong coupling constant α_s is tiny. This corresponds to a temperature which is several orders of magnitude larger than those relevant for experiments at RHIC and LHC.

The poor convergence of weak-coupling expansion also shows up in the case of scalar field theory. For a massless scalar field theory with a $g^2\phi^4/4!$ interaction, the weak-coupling expansion for the pressure to order g^5 is ¹⁻³

$$\mathcal{P} = \mathcal{P}_{\text{ideal}} \left[1 - \frac{5}{4}\alpha + \frac{5\sqrt{6}}{3}\alpha^{3/2} + \frac{15}{4} \left(\log \frac{\mu}{2\pi T} + 0.40 \right) \alpha^2 - \frac{15\sqrt{6}}{2} \left(\log \frac{\mu}{2\pi T} - \frac{2}{3} \log \alpha - 0.72 \right) \alpha^{5/2} + \mathcal{O}(\alpha^3 \log \alpha) \right], \quad (1.1)$$

where $\mathcal{P}_{\text{ideal}} = (\pi^2/90)T^4$ is the pressure of the ideal gas of a free massless boson, $\alpha = g^2(\mu)/(4\pi)^2$, and $g(\mu)$ is the $\overline{\text{MS}}$ coupling constant at the renormalization scale μ . In Fig. 1, we show the successive perturbative approximations to $\mathcal{P}/\mathcal{P}_{\text{ideal}}$ as a function of $g(2\pi T)$. Each partial sum is shown as an

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error band obtained by varying μ from πT to $4\pi T$. To express $g(\mu)$ in terms of $g(2\pi T)$, we use the numerical solution to the renormalization group equation $\mu \frac{\partial}{\partial \mu} \alpha = \beta(\alpha)$ with a five-loop beta function. The lack of convergence of the

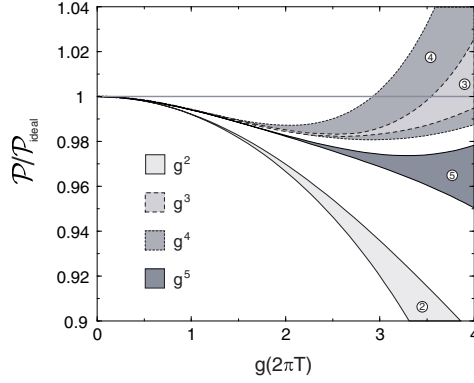


Figure 1: Weak-coupling expansion to orders g^2 , g^3 , g^4 , and g^5 for the pressure normalized to that of an ideal gas as a function of $g(2\pi T)$.

perturbative series is evident in Fig. 1. The band obtained by varying μ by a factor of two is a lower bound on the theoretical error involved in the calculations. Another indicator of the error is the difference between successive approximations. From Fig. 1, we conclude that the error grows quickly for $g \geq 1.5$.

2 Screened Perturbation Theory

Screened perturbation theory, which was introduced by Karsch, Patkós and Petreczky⁴, is simply a reorganization of the perturbation series for thermal field theory.

The Lagrangian density for a massless scalar field with a ϕ^4 interaction is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{24} g^2 \phi^4 + \Delta \mathcal{L} , \quad (2.2)$$

where g is the coupling constant and $\Delta \mathcal{L}$ includes counterterms. The Lagrangian density is written as

$$\mathcal{L}_{\text{SPT}} = -\mathcal{E}_0 + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} (m^2 - m_1^2) \phi^2 - \frac{1}{24} g^2 \phi^4 + \Delta \mathcal{L} + \Delta \mathcal{L}_{\text{SPT}} , \quad (2.3)$$

where \mathcal{E}_0 is a vacuum energy density parameter and we have added and subtracted mass terms. If we set $\mathcal{E}_0 = 0$ and $m_1^2 = m^2$, we recover the original

Lagrangian (2.2). Screened perturbation theory is defined by taking m^2 to be of order g^0 and m_1^2 to be of order g^2 , expanding systematically in powers of g^2 , and setting $m_1^2 = m^2$ at the end of the calculation. This defines a reorganization of perturbation theory in which the expansion is around the free field theory defined by

$$\mathcal{L}_{\text{free}} = -\mathcal{E}_0 + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \quad (2.4)$$

The interaction term is

$$\mathcal{L}_{\text{int}} = -\frac{1}{24}g^2\phi^4 + \frac{1}{2}m_1^2\phi^2 + \Delta\mathcal{L} + \Delta\mathcal{L}_{\text{SPT}}. \quad (2.5)$$

At each order in screened perturbation theory, the effects of the m^2 term in (2.4) are included to all orders. However when we set $m_1^2 = m^2$, the dependence on m is systematically subtracted out at higher orders in perturbation theory by the m_1^2 term in (2.5). At nonzero temperature, screened perturbation theory does not generate any infrared divergences, because the mass parameter m^2 in the free Lagrangian (2.4) provides an infrared cutoff. The resulting perturbative expansion is therefore a power series in g^2 and $m_1^2 = m^2$ whose coefficients depend on the mass parameter m .

This reorganization of perturbation theory generates new ultraviolet divergences, but they can be canceled by the additional counterterms in $\Delta\mathcal{L}_{\text{SPT}}$. The renormalizability of the Lagrangian in (2.3) guarantees that the only counterterms required are proportional to 1, ϕ^2 , $\partial_\mu\phi\partial^\mu\phi$, and ϕ^4 .

2.1 Mass Prescriptions

At this point I would like to emphasize that the mass parameter in SPT is completely arbitrary, and we need a prescription for it. The prescription of Karsch, Patkós, and Petreczky for $m_*(T)$ is the solution to the “one-loop gap equation”:

$$m_*^2 = 4\alpha(\mu_*) \left[\int_0^\infty dk \frac{k^2}{\omega(e^{\beta\omega} - 1)} - \frac{1}{8} \left(2 \log \frac{\mu_*}{m_*} + 1 \right) m_*^2 \right], \quad (2.6)$$

where $\omega = \sqrt{k^2 + m_*^2}$ and $\alpha(\mu_*) = g^2(\mu_*)/(4\pi)^2$.

There are many possibilities for generalizing (2.6) to higher orders in g . Here, I consider three generalizations.

- The *screening mass* m_s is defined by the location of the pole of the static propagator:

$$\mathbf{p}^2 + m^2 + \Pi(0, \mathbf{p}) = 0, \quad \text{at} \quad \mathbf{p}^2 = -m_s^2. \quad (2.7)$$

- The *tadpole mass* m_t is defined by the expectation value of ϕ^2 :

$$m_t^2 = g^2 \langle \phi^2 \rangle . \quad (2.8)$$

- The *variational mass* m_v is the solution to

$$\frac{d}{dm^2} \mathcal{F}(T, g(\mu), m, m_1 = m, \mu) = 0 \quad (2.9)$$

Thus the dependence of the free energy on m is minimized by m_v .

The above mass prescriptions all coincide at the one-loop order and is given by (2.6) above. The three masses differ at the two-loop level and beyond. The two-loop gap equation for the tadpole mass turns out to be identical to the one-loop gap equation. Note that the tadpole mass cannot be generalized to gauge theories since the expectation value $\langle A_\mu A^\mu \rangle$ is a gauge-variant quantity. Moreover, the screening mass in nonabelian gauge theories is not defined beyond leading order in perturbation theory due to a logarithmic infrared divergence.

2.2 Results

A thorough study of screened perturbation theory is presented in Ref.⁵. Here, I only present a few selected results.

In Fig. 2, we show the one-, two- and three-loop SPT-improved approximations to the pressure using the tadpole gap equation. The bands are obtained by varying μ by a factor of two around the central values $\mu = 2\pi T$ and $\mu = m_*$. The choice $\mu = m_*$ gives smaller bands from varying the renormalization scale, but this is mainly due to the fact that $g(2\pi T)$ is larger than $g(m_*)$. The vertical scale in Fig. 2 has been expanded by a factor of about two compared to Fig. 1, which shows the successive approximations using the weak-coupling expansion. All the bands in Fig. 2 lie within the g^5 band in Fig. 1. Thus we see a dramatic improvement in the apparent convergence compared to the weak-coupling expansion.

3 Summary

In this talk, I have briefly discussed SPT, which is a reorganization of the perturbative expansion. In contrast to the weak-coupling expansion, the SPT-improved approximations to pressure appear to converge for rather large values of the coupling constant.

Screened perturbation theory has been generalized to gauge theories and is called hard-thermal-loop (HTL) perturbation theory⁶. A one-loop calculation

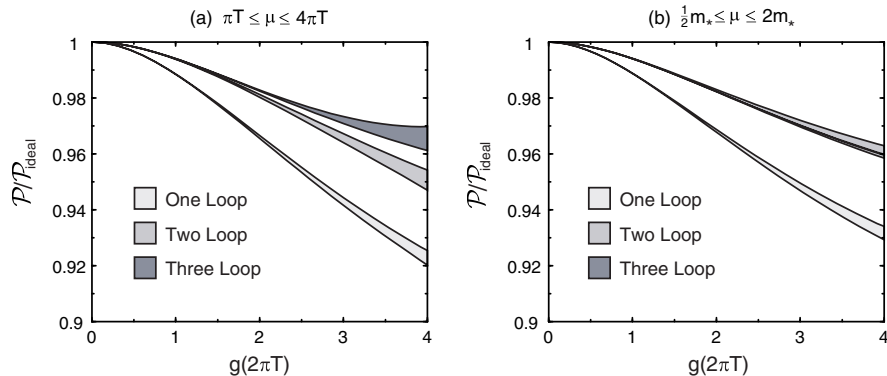


Figure 2: One-, two-, and three-loop SPT-improved pressure as a function of $g(2\pi T)$ for (a) $\pi T < \mu < 4\pi T$ and (b) $\frac{1}{2}m_* < \mu < 2m_*$.

of the pressure with and without fermions has already been carried out⁶. Two-loop calculations are in progress⁷.

The fact that SPT shows very good convergence properties gives us hope that HTL perturbation theory will be a consistent approach that can be used for calculating static and dynamical quantities of a quark-gluon plasma.

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